THE TECHNOLOGICAL INSTITUTE

has been approved

24

NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS

Best
Available
Copy

SYSTEMS RESEARCH MEMORANDUM No. 216

The Technological Institute The College of Arts and Sciences

Northwestern University

A DECOMPOSITION METHOD FOR INTERVAL LINEAR PROGRAMMING

by

P. D. Robers*
A. Ben-Israel**

August 1968

*The contribution of Philip D. Robers is part of his Ph. D. dissertation in Operations Research at Northwestern University, August 1968.

**This research was partly supported by the National Science Foundation, Project GP 7550, and by the Office of Naval Research, Contract Nonr-1228(10) N00014-67-A-0356-0006, and by the U.S. Army Research Office- Durham, Contract No. DA-31-124-ARO-D-322, at Northwestern University.

SYSTEMS RESEARCH GROUP

A. Charnes, Director

0. Introduction

An <u>interval linear program</u> is any problem of the form

$$
\begin{aligned}
&\text{(IP)} : \text{maximize } c^t x \\
&\text{subject to} \\
&b^- \leq A x \leq b^+
\end{aligned}
$$

where the matrix A, vectors b^-, b^+, c are given. A finite iterative algorithm for solving (IP) is given here which is based on the Dantzig-Wolfe decomposition principle [6]. An alternative iterative method for solving (IP) is given in [8] and [9].

(IP) can always be solved by the standard linear programming techniques (e.g. [4], [5]) by treating each two-sided constraint as two constraints and expressing each unrestricted variable as the difference of two nonnegative variables. Since this procedure increases considerably the effective problem size and since problems of form (IP) are common in many applications (e.g. in structural analysis, production planning, approximation problems, etc.; see [2], [3], and [8]), it is desirable to have special techniques for solving (IP). Furthermore, (IP) is a sufficiently general model to cover all bounded linear programs so that any method for solving (IP) may be viewed as an alternative method for linear programming. In particular, a problem in the standard form

$$
\begin{aligned}
&\text{maximize } c^t x \\
&\text{subject to} \\
&A x \leq b \\
&\quad x \geq 0 ,
\end{aligned}
$$

if bounded, is equivalent to the (IP)

$$\begin{aligned} &\text{maximize } c^t x \\ &\text{s.t.} \\ &-Me \leqq Ax \leqq b \\ &0 \leqq x \leqq Me \end{aligned}$$

where e is a vector of ones and M is a sufficiently large positive scalar.

Notations and preliminaries are given in section 1. In section 2 it is demonstrated that any (IP) can be converted to decomposable form so that the decomposition method, developed in section 3, can be applied. A numerical example is given at the end of section 3.

1. Notations and Preliminaries

The following notations are used:

iff — if and only if;

s.t. — subject to;

$\{x : f(x)\}$ — the set of x satisfying $f(x)$;

$\emptyset$ — the empty set ;

R^n — the n-dimensional real vector space;

$R^{m \times n}$ — the space of $m \times n$ real matrices;

$R_r^{m \times n}$ — the set $\{X \in R^{m \times n} : \text{rank } X = r\}$;

I_n — the $n \times n$ identity matrix;

e_i — the $i^{\underline{th}}$ column of I_n ;

e — the vector of ones;
(the dimension of e and e_i will be clear from context)

For any $x, y \in R^n$:

$x \geqq y$ — denotes $x_i \geq y_i$ for $i = 1, \ldots, n$;

$x \perp y$ denotes x is perpendicular to y (i.e. $\Sigma x_i y_i = 0$).

For any subspace L of R^n:

$x + L$ denotes the <u>manifold</u> $\{x + \ell : \ell \in L\}$;

For any $A \in R^{m \times n}$ denote by:

A^t the <u>transpose</u> of A;

$R(A)$ the <u>range space</u> of A (i.e. $\{y \in R^m : y = Ax$ for some $x \in R^n\}$);

$N(A)$ the <u>null space</u> of A (i.e. $\{x \in R^n : Ax = 0\}$);

A^{-1} the <u>inverse</u> of A, if nonsingular.

An <u>interval linear program</u> or <u>interval linear programming problem</u> is any problem of the form

(1) $$(\text{IP}): \text{maximize } c^t x$$

s.t.

(2) $$b^- \leqq Ax \leqq b^+$$

where $c = (c_i)$, $b^- = (b_j^-)$, $b^+ = (b_j^+)$, and $A = (a_{ij})$ $(i = 1, \ldots, n : j = 1, \ldots, m)$ are given, with

(3) $$b^- \leqq b^+ .$$

Let S be the set of points satisfying (2)

(4) $$S = \{x \in R^n : b^- \leqq Ax \leqq b^+\}.$$

A point $x \in S$ is called a <u>feasible solution</u> of (IP). Problem (IP) is <u>feasible</u> if $S \neq \emptyset$, otherwise <u>infeasible.</u> If $A \in R_m^{m \times n}$ (IP) is feasible for any $b^- \leqq b^+$ since $R(A) = R^m$. (IP) is <u>bounded</u> if $S \neq \emptyset$ and

and $\max_{x \in S} c^t x < \infty$, otherwise <u>unbounded</u>. Clearly (IP) is bounded if S is a bounded set: the converse is false. Boundedness is characterized in:

<u>Lemma 1</u> ([1]): <u>Let</u> $S \neq \emptyset$. <u>Then</u> (IP) <u>is bounded iff</u>

(5) $$c \perp N(A) . \qquad \square$$

We conclude that all bounded (IP) problems have $c \perp N(A)$ and consequently any (IP) with $c \not\perp N(A)$ is uninteresting .

Condition (5) may be verified if a basis for $N(A)$ is available. The following recipe may be used to compute such a basis:

<u>Lemma 2</u> ([1]): <u>Let</u> $A \in R^{m \times n}$ <u>and</u> $E \in R_m^{m \times m}$ <u>satisfy</u>

(6) $$EA = \begin{pmatrix} I_r & \Delta \\ \hline \multicolumn{2}{c}{O_{(m-r) \times n}} \end{pmatrix} P$$

<u>where</u> P <u>is a permutation matrix.</u> <u>Then</u> rank $(A) = r$ <u>and the columns of</u>

(7) $$N = P^t \begin{pmatrix} -\Delta \\ I_{n-r} \end{pmatrix}$$

<u>form a basis of</u> $N(A)$. $\square$

The following example will demonstrate the use of lemma 2:

<u>Example 1:</u> Find N for the matrix

$$A = \begin{pmatrix} 1 & 4 & 2 & 0 \\ -1 & 2 & 0 & -2 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix} .$$

We diagonalize A as shown in the following table (pivot elements are circled):

$$A^{(o)} = \begin{pmatrix} 1 & 4 & 2 & 0 \\ -1 & 2 & 0 & -2 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & 6 & 2 & -2 \\ 0 & -6 & -2 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 1 & 0 & 2/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4/3 & 4/3 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$A^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(interchanging rows).}$$

From $A^{(4)} = \begin{pmatrix} I_3 & \Delta \\ \hline \multicolumn{2}{c}{O_{14}} \end{pmatrix} P$ we see that

rank $(A) = 3$, $\Delta = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ and $N = \begin{pmatrix} -\Delta \\ I_1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. □

Computationally, lemma 2 is a very naive statement, but we will ignore the computational difficulties of ill-conditioning for which the reader is referred to [10] .

It will now be shown that any (IP) with $c \perp N(A)$ can be converted to an equivalent (IP) having full column rank (i.e. with $A \in R_r^{m \times r}$).

<u>Lemma 3</u>: <u>Let</u> $A \in R_r^{m \times n}$ <u>and</u> $D \in R_r^{r \times n}$ <u>such that</u>

(8) $$R(D^t) = R(A^t) .$$

Then

(9) $$AD^t \in R_r^{m\times r} \ .$$

Proof: Obvious. □

Lemma 4: Suppose that (IP) is given with $A \in R_r^{m\times n}$, $c \perp N(A)$ and let D be given as in lemma 3. Then the optimal solutions of (IP) are

(10) $$D^t y^* + N(A)$$

where y^* is any optimal solution of

(11) $$\text{maximize} \quad c^t D^t y$$

$$\text{s.t.}$$

(12) $$b^- \leqq A D^t y \leqq b^+ \ .$$

Proof: Using the facts that $c \perp N(A)$ and $R(A^t) = N(A)^\perp$ (e.g. [7]) it follows that the optimal solutions of (IP) are

(13) $$x^* + N(A)$$

where x^* is any optimal solution of

(1) $$\text{maximize} \quad c^t x$$

$$\text{s.t.}$$

(2) $$b^- \leqq A x \leqq b^+$$

(14) $$x \in R(A^t) \ .$$

From (8) it follows that any $x \in R(A^t)$ can be written

(15) $$x = D^t y$$

for some $y \in R^r$. Substituting (15) in (13) and in problem (1), (2), (14) gives the desired result. □

In what follows, it is assumed that lemma 4 has been applied, if necessary, so that the problem to be solved has a coefficient matrix of full column rank. This assumption is not essential, but it simplifies notations, permits the use of ordinary (rather than generalized) inverses,

and should reduce computational effort if the original problem did not have full column rank.

The following elementary result is a special case of the result given in [1] and will be used to develop the algorithm below:

<u>Lemma 5:</u> <u>Let</u> (IP) <u>be given with</u> $A \in R^{n\times n}_{n}$ <u>and</u> (3). <u>Then the optimal solutions of</u> (1), (2) <u>are all the vectors of the form</u>

(16) $$x^* = A^{-1} z^*$$

<u>where the components of</u> $z^* \in R^n$ <u>are defined by</u>

(17) $$z_j^* = \begin{cases} b_j & \\ \theta_j a_j + (1 - \theta_j) b_j & \text{if } (c^t A^{-1})_j \\ a_j & \end{cases} \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

<u>for</u> $j = 1,\dots,n$ <u>and</u> $0 \le \theta_j \le 1$ <u>for those</u> j <u>with</u> $(c^t A^{-1})_j = 0$.

<u>Proof:</u> Substituting

(18) $$z = Ax$$

in (1), (2) we obtain the equivalent problem

(19) $$\text{maximize } c^t A^{-1} z$$

s.t.

(20) $$b^- \leqq z \leqq b^+$$

whose optimal solutions are the vectors z^* given by (17). The reverse substitution gives (16). $\square$

Conditions for the explicit solution of the general IP were given in [11].

2. <u>Converting (IP) to Decomposable Form</u>

A <u>decomposable interval program</u> is any problem having the form

(1) $$\text{(DIP): maximize } c^t x$$

s.t.

(21) $$\bar{b}^- \leqq \bar{A}x \leqq \bar{b}^+$$

$$(22) \qquad \hat{b}^- \leqq \hat{A}x \leqq \hat{b}^+$$

where $\bar{A}$ and $\hat{A}$ are nonsingular (say $\bar{A}$, $\hat{A} \in R_p^{pxp}$), $\bar{b}^- \leqq \bar{b}^+$, and $\hat{b}^- \leqq \hat{b}^+$. The method developed in section 3 below is applicable to any (DIP) hence to any bounded (IP), since any bounded (IP) can be converted to an equivalent (DIP), as we now demonstrate:

Consider the (IP):

$$(1) \qquad \text{maximize } c^t x$$

s. t.

$$(2) \qquad b^- \leqq Ax \leqq b^+$$

where (3) is satisfied and $A \in R_r^{mxr}$ (e. g. lemma 4). This problem is bounded since $N(A) = \{0\}$.

Rearrange the constraints as necessary to put (1), (2) in the form

$$(1) \qquad \text{maximize } c^t x$$

subject to

$$(23) \qquad b_1^- \leqq A_1 x \leqq b_1^+$$

$$(24) \qquad b_2^- \leqq A_2 x \leqq b_2^+$$

$$(25) \qquad b_3^- \leqq A_3 x \leqq b_3^+$$

where $A_1 \in R_r^{rxr}$, $A_2 \in R_q^{qxr}$, and $A_3 \in R^{(m-r-q)xr}$. That is, A_1 is a nonsingular submatrix of A, A_2 is any submatrix having full row rank whose rows are not in A_1 , and A_3 is made up of the rows of A not in A_1 or A_2 . Note that q is not uniquely defined, but as we shall see later it is desirable to make q as large as possible ($q = 0$ is always possible).

Clearly we can always choose

$$(26) \qquad b_4^- \leqq Bx \leqq b_4^+ \ ,$$

a subset of the constraints (23)

such that $\begin{pmatrix} A_2 \\ B \end{pmatrix} \in R_r^{r\times r}$. Problem (IP) is not changed by including some constraints more than once, so (1), (2) may be written

(1) maximize $c^t x$

subject to

(23) $b_1^- \leqq A_1 x \leqq b_1^+$

(24) $b_2^- \leqq A_2 x \leqq b_2^+$

(26) $b_4^- \leqq Bx \leqq b_4^+$

(25) $b_3^- \leqq A_3 x \leqq b_3^+$.

Finally, observe that x^* is an optimal solution to (IP) iff (x^*, y^*) is an optimal solution to the following bounded problem which clearly has form (DIP):

(1) maximize $c^t x$

subject to

(27) $$\begin{pmatrix} b_1^- \\ 0 \end{pmatrix} \leqq \begin{pmatrix} A_1 & 0 \\ 0 & I_{m-r-q} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leqq \begin{pmatrix} b_1^+ \\ 0 \end{pmatrix}$$

(28) $$\begin{pmatrix} b_2^- \\ b_4^- \\ b_3^- \end{pmatrix} \leqq \begin{pmatrix} A_2 & 0 \\ B & 0 \\ A_3 & I_{m-r-q} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leqq \begin{pmatrix} b_2^+ \\ b_4^+ \\ b_3^+ \end{pmatrix} .$$

A procedure for identifying appropriate A_1, A_2, A_3, and B is demonstrated in the following trivial example:

<u>Example 2:</u> Transform the following (IP) problem to form (DIP):

maximize $x_1 + x_2$

subject to

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \leq \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 1 \\ -1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

Applying Gauss-Jordan eliminations to A we obtain (pivots circled):

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 1 \\ -1 & -1 \\ 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since rows 1 and 3 contained pivots we conclude that

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a suitable nonsingular submatrix of A.

Rearranging A with matrix A_1 at the bottom, we repeat the above proceedure:

$$\begin{pmatrix} 2 & 2 \\ -1 & -1 \\ 2 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

We conclude that row 4 of A is linearly dependent on row 2 (since all its elements vanished after step 1) so that row 4 must fall in A_3. Furthermore since $r = 2$ pivots were found before reaching the bottom r rows we conclude that $B = 0$, and

$$A_2 = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$$

since pivots were contained in rows 1 and 3 (i.e. rows 2 and 5 of the original matrix).

The equivalent (DIP) is therefore:

maximize $x_1 + x_2$

subject to

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \leq \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \leq \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix}$$

3. The Decomposition Algorithm

Problem (DIP) : (1), (21), (22) may be equivalently written

maximize $c^t \bar{x}$

subject to

(29) $\bar{x} = \hat{x}$

$$\bar{b}^- \leq \bar{A}\bar{x} \leq \bar{b}^+$$

$$\hat{b}^- \leq \hat{A}\hat{x} \leq \hat{b}^+ \quad .$$

In this section we develop an algorithm, for solving problem (29) which is related to the Dantzig-Wolfe decomposition principle [6] .

Let

(30) $\bar{S} = \{x \in R^p : \bar{b}^- \leq \bar{A}x \leq \bar{b}^+ \}$

and let $\bar{G}$ be the finite matrix whose columns are the extreme points of

$\bar{S}$. Since $\bar{A}$ is nonsingular, $\bar{S}$ is a bounded polyhedron so that $\bar{x} \in \bar{S}$ iff

(31) $$\bar{x} = \bar{G}\,\bar{v}\,,\quad e^t\bar{v} = 1\quad,\quad \bar{v} \geqq 0$$

(i.e. $\bar{x}$ is a convex combination of the extreme points of $\bar{S}$). An analagous result holds for

(32) $$\hat{S} = \{x \in R^p : \hat{b}^- \leqq \hat{A}x \leqq \hat{b}^+\}$$

and the corresponding matrix $\hat{G}$ whose columns are the extreme points of $\hat{S}$.

Consequently, (29) may be written

$$\text{maximize}\quad c^t\,\bar{G}\,\bar{v}$$

subject to

(31) $$\begin{aligned} \bar{G}\,\bar{v} - \hat{G}\hat{v} &= 0 \\ e^t\bar{v} &= 1 \\ e^t\hat{v} &= 1 \\ \bar{v}, \hat{v} &\geqq 0 \end{aligned}$$

Problem (31) has the "standard" linear programming form except that the columns of $\bar{G}$ and $\hat{G}$ are not immediately known. Indeed, we shall solve (31) using the simplex algorithm (e.g. [4], [5]) with a special technique for generating the columns of $\bar{G}$ and $\hat{G}$ one at a time as needed.

Suppose that a basic feasible solution to (31) is in hand with "simplex multipliers" (e.g. [5]).

(32) $$(\pi_1, \ldots, \pi_p\,,\ \sigma_1\,,\ \sigma_2) = (\pi,\ \sigma_1\,,\ \sigma_2)$$

Let the columns of $\bar{G}$ and $\hat{G}$ be denoted by $\bar{g}_i$ $(i = 1, \ldots, \bar{N})$ and $\hat{g}_i$ $(i = 1, \ldots, \hat{N})$ respectively. A vector

$$\begin{pmatrix} \bar{g}_i \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\hat{g}_i \\ 0 \\ 1 \end{pmatrix}$$

can enter the basis if

(33) $$(\pi, \sigma_1, \sigma_2)\begin{pmatrix}\bar{g}_i\\1\\0\end{pmatrix} - c^t\bar{g}_i = (\pi - c^t)\bar{g}_i + \sigma_1 < 0$$

or

(34) $$(\pi, \sigma_1, \sigma_2)\begin{pmatrix}-\hat{g}_i\\0\\1\end{pmatrix} = -\pi\hat{g}_i + \sigma_2 < 0$$

respectively (i. e. it has a negative relative cost).

Following the standard simplex proceedure we bring into the basis the vector with the smallest relative cost

(35) $$\lambda = \min\{\min_{\bar{g}_i}((\pi - c^t)\bar{g}_i + \sigma_1) \quad , \quad \min_{\hat{g}_i}(-\pi\hat{g}_i + \sigma_2)\}$$

unless $\lambda \geq 0$ which indicates the present solution is optimal. Thus we must determine the extreme points $\bar{g}^*$ and $\hat{g}^*$ such that

(36) $$(\pi - c^t)\bar{g}^* = \min_{\bar{g}_i}((\pi - c^t)\bar{g}_i)$$

and

(37) $$-\pi\hat{g}^* = \min_{\hat{g}_i}(-\pi\hat{g}_i) \,.$$

But $\min_{\bar{g}_i}((\pi - c^t)\bar{g}_i) = \min_{x\in\bar{S}}((\pi - c^t)x)$ so that (by lemma 5)

(38) $$\bar{g}^* = \sum_{i\in\bar{I}_+}\bar{b}_i^+\bar{t}_i + \sum_{i\in\bar{I}_-}\bar{b}_i^-\bar{t}_i$$

(39) where $\bar{A}^{-1} = (\bar{t}_1, \ldots, \bar{t}_p)$

(40) and $\bar{I}_{+,-} = \{i : (\pi - c^t)\bar{t}_i \geq, < 0\}$

(i. e. $\bar{g}^*$ is an extreme point solution of the subproblem

$\min_{x\in\bar{S}}(\pi - c^t)x$.)

Likewise

(41) $$\hat{g}^* = \sum_{i \in \hat{I}_+} \hat{b}_i^+ \hat{t}_i + \sum_{i \in \hat{I}_-} \hat{b}_i^- \hat{t}_i$$

(42) where $\hat{A}^{-1} = (\hat{t}_1, \ldots, \hat{t}_p)$

(43) and $\hat{I}_{+,-} = \{ i : -\pi \hat{t}_i \geq, < 0 \}$.

If $\lambda < 0$ either $\begin{pmatrix} \bar{g}^* \\ 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} \hat{g}^* \\ 0 \\ 1 \end{pmatrix}$

enters the basis depending on which has the lowest relative cost. The simplex iteration is then completed to obtain a "better" basic feasible solution and the entire procedure is repeated. The theory of the simplex method assures finiteness provided proper steps are taken to treat degeneracy.

If $\lambda \geq 0$ the present solution, call it $(\bar{v}^*, \hat{v}^*)$ to (31) is optimal. An optimal solution to (DIP) : (1), (21), (22) is then

(44) $$x^* = \sum_{i \in \Omega_1} \bar{g}_i \bar{v}_i^* = \sum_{i \in \Omega_2} \hat{g}_i \hat{v}_i^*$$

where $\Omega_1 = \{ i : \bar{v}_i > 0 \}$

$\Omega_2 = \{ i : \hat{v}_i > 0 \}$.

An initial basic feasible solution can be found using artificial variables. One approach is to begin with the enlarged problem

maximize $c^t \bar{G} \bar{v} - M e^t z$

(45) subject to

$$\begin{pmatrix} \bar{G} & -\hat{G} & \\ e^t & 0 & I_{p+2} \\ 0 & e^t & \end{pmatrix} \begin{pmatrix} \bar{v} \\ \hat{v} \\ z \end{pmatrix} = \begin{pmatrix} 0_{p\times 1} \\ 1 \\ 1 \end{pmatrix}$$

$$\bar{v}, \hat{v}, z \geqq 0$$

where M is a sufficiently large scalar, using $\{e_1, \ldots, e_{p+2}\}$ as the starting basis. If $\lambda \geq 0$ while some of the artificial variables (i.e. $\{z_i, i = 1, \ldots, p+2\}$) are still in the basis at a non-zero level, problem (DIP) has no feasible solutions.

A slightly different approach for obtaining an initial basic feasible solution to (31) is to find any extreme points of $\bar{S}$ and $\hat{S}$. If we call these points $\bar{g}$, and $\hat{g}$, respectively[1], then

(46) $$\begin{pmatrix} \bar{g}_1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\hat{g}_1 \\ 0 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} e_1 \\ 0 \\ 0 \end{pmatrix}, \ldots, \pm \begin{pmatrix} e_p \\ 0 \\ 0 \end{pmatrix}$$

is a suitable starting basis to the equivalent problem

(47) $$\text{maximize} \quad c^t \bar{G} \bar{v} - M e^t z$$

subject to

$$\begin{pmatrix} \bar{G} & -\hat{G} & I_p \\ e^t & 0 & 0 \\ 0 & e^t & 0 \end{pmatrix} \begin{pmatrix} \bar{v} \\ \hat{v} \\ z \end{pmatrix} = \begin{pmatrix} 0_{p\times 1} \\ 1 \\ 1 \end{pmatrix}$$

$$\bar{v}, \hat{v}, z \geq 0$$

The sign in front of $\begin{pmatrix} e_i \\ 0 \\ 0 \end{pmatrix}$ in (46) will depend on the sign of $(\bar{g}_{1i} - \hat{g}_{1i})$ and has purposely been left ambiguous to simplify notation. An example should clarify the use of (47).

<u>Example 3:</u> maximize $x_1 + 2x_2$

subject to

(48) $$\begin{aligned} 0 \leq x_1 \quad\quad\; &\leq 6 \\ 0 \leq \quad\quad x_2 &\leq 8 \\ 2 \leq x_1 + x_2 &\leq 6 \\ -9 \leq -3x_1 + x_2 &\leq 9 \end{aligned}$$

[1] Heuristically, a good choice for $\bar{g}$, $\hat{g}$, would seem to be optimal extreme point solutions to $\max_{x \in \bar{S}} c^t x$ and $\max_{x \in \hat{S}} c^t x$ respectively.

This problem has the form of (DIP) : (1), (21), (22) with

$$\bar{b}^- = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \bar{A}^{-1}, \quad \bar{b}^+ = \begin{pmatrix} 6 \\ 8 \end{pmatrix},$$

$$\hat{b}^- = \begin{pmatrix} 2 \\ -9 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}, \quad \hat{A}^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \quad \hat{b}^+ = \begin{pmatrix} 6 \\ 9 \end{pmatrix},$$

$$\bar{S} = \{x \in R^2 : 0 \le x_1 \le 6 \ , \ 0 \le x_2 \le 8\}$$

$$\hat{S} = \{x \in R^2 : 2 \le x_1 + x_2 \le 6 \ , \ -9 \le -3x_1 + x_2 \le 9\} \ .$$

We work with the following problem (see (47) which is equivalent to (48):

$$\text{maximize} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}^t \left(\sum_{i=1}^{\bar{N}} \bar{g}_i \bar{v}_i \right) - M (z_1 + z_2)$$

subject to

$$\left(\begin{array}{c:c:cc} \bar{g}_1 \ldots \bar{g}_{\bar{N}} & -\hat{g}_1 \ldots -\hat{g}_{\hat{N}} & -1 & 0 \\ & & 0 & -1 \\ \hdashline 1 \ldots 1 & 0 \ldots 0 & 0 & 0 \\ 0 \ldots 0 & 1 \ldots 1 & 0 & 0 \end{array}\right) \begin{pmatrix} \bar{v} \\ \hat{v} \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{49}$$

$$\bar{v} \ , \ \hat{v} \ , \ z_1 \ , \ z_2 \ge 0$$

<u>Iteration 0:</u>

$$\bar{g}_1 = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \quad \text{and} \quad \hat{g}_1 = \begin{pmatrix} -\frac{3}{4} \\ \frac{27}{4} \end{pmatrix} \quad \text{are}$$

optimal extreme point solutions to the subproblems

$$\max_{x \in \bar{S}} \ x_1 + 2x_2$$

and

$$\max_{x \in \hat{S}} \ x_1 + 2x_2$$

respectively. Then the columns of

$$B = \begin{pmatrix} 6 & \frac{3}{4} & -1 & 0 \\ 8 & -\frac{27}{4} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

form a basis which gives the following initial basic feasible solution to problem (49): $\bar{v}_1 = 1$, $\hat{v}_1 = 1$, $z_1 = (6 + \frac{3}{4}) = \frac{27}{4}$ and $z_2 = (8 - \frac{27}{4}) = \frac{5}{4}$.

Since

$$B^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 6 & \frac{3}{4} \\ 0 & -1 & 8 & -\frac{27}{4} \end{pmatrix} ,$$

the corresponding simplex multipliers are

$(\pi, \sigma_1 , \sigma_2) = (22, 0, -M , -M)B^{-1} = (M , M , 22 - 14M , 6M)$.

__Iteration 1:__

Since $\pi - c^t = (M - 1, M - 2)$, (38) and (41) give

$$\bar{g}^* = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

$$\hat{g}^* = 6 \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} + 9 \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{15}{4} \\ \frac{27}{4} \end{pmatrix} .$$

The corresponding relative costs ((33) and (34)) are

$$(\pi - c^t) \bar{g}^* + \sigma_1 = -14M + 22$$

$$-\pi \hat{g}^* + \sigma_2 = -\frac{19}{4} M .$$

Thus $\begin{pmatrix} \bar{g}_2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ enters the basis, and the reader can check that $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ leaves. The new basis vectors are the

columns of
$$B = \begin{pmatrix} 6 & \frac{3}{4} & -1 & 0 \\ 8 & -\frac{27}{4} & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and the new basic feasible solution is

$$\bar{v}_1 = \frac{27}{32}, \quad \bar{v}_2 = \frac{5}{32}, \quad \hat{v}_1 = 1, \quad z_1 = \frac{13}{16} \quad .$$

The new simplex multipliers are

$$\left(M, \ \frac{22}{8} - \frac{3}{4} M, \ 0, \ \frac{501}{16} - \frac{93}{16} M \right).$$

<u>Iteration 2:</u> In a similar manner, the reader can check that

$$\begin{pmatrix} \bar{g}_2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 1 \\ 0 \end{pmatrix} \text{ enters the basis and } \begin{pmatrix} \bar{g}_1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 1 \\ 0 \end{pmatrix} \text{ leaves.}$$

The new basic feasible solution is

$\bar{v}_2 = \frac{5}{32}, \quad \bar{v}_3 = \frac{27}{32}, \quad \hat{v}_1 = 1, \quad z_1 = \frac{3}{4}.$ The basis matrix is

$$B = \begin{pmatrix} 0 & \frac{3}{4} & -1 & 0 \\ 8 & -\frac{27}{4} & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and the simplex multipliers are

$$(16, \ 0, \ -M, \ 0)B^{-1} = \left(-M, \ 2, \ 0, \ -\frac{3}{4} M + \frac{27}{2}\right).$$

<u>Iteration 3:</u>

$$\begin{pmatrix} -\hat{g}_2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{15}{4} \\ -\frac{9}{4} \\ 0 \\ 1 \end{pmatrix} \quad \text{enters the basis}$$

and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ leaves so that the new basic feasible solution

is

$$\bar{v}_2 = \frac{1}{4}, \bar{v}_3 = \frac{3}{4}, \hat{v}_1 = \frac{5}{6}, \hat{v}_2 = \frac{1}{6} .$$

The simplex multipliers are

$$(2, 2, 0, 12) .$$

<u>Iteration 4:</u>

$$\pi - c^t = (2,2) - (1,2) = (1,0)$$

so that from (18) and (21), $\bar{g}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\hat{g}^* = \begin{pmatrix} \frac{15}{4} \\ \frac{9}{4} \end{pmatrix}$. From (35), $\lambda = 0$ so that the optimal basic feasible solution is (from (44))

$$x^* = \frac{3}{4} \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} .$$

References

[1] Ben-Israel, A., and A. Charnes, "An explicit solution of a special class of linear programming problems," Operations Research, (forthcoming).

[2] Ben-Israel, A., A. Charnes, A. P. Hurter, P. D. Robers, "On the explicit solution of a special class of linear economic models," Systems Research Memo. No. 215, The Technological Institute, Northwestern University, August 1968.

[3] Ben-Israel, A., A. Charnes, and P. D. Robers, "On generalized inverses and interval linear programming," Proceedings of the Symposium 'Theory and applications of generalized inverses', Texas Technological College, Lubbock, Texas, March 1968, (forthcoming).

[4] Charnes, A., and W. W. Cooper, Management Models and Industrial Applications of Linear Programming, vol. I, II, John Wiley & Sons, Inc., New York, 1961.

[5] Dantzig, G. B., Linear Programming and Extensions, Princeton, 1963.

[6] Dantzig, G. B. and P. Wolfe, "The decomposition principle for linear programs," Operations Research, 8 (1960), pp. 101-11.

[7] Halmos, P. R., Finite Dimensional Vector Spaces, Second Edition, Van Norstrand, Princeton, 1958.

[8] Robers, P. D., "Interval linear programming," Ph. D. thesis, Dept. of Industrial Engineering and Management Sciences, Northwestern University, August 1968.

[9] Robers, P. D., and A. Ben-Israel, "A suboptimization method for interval linear programming: A new method for linear programming," Systems Research Memo. No. 206, The Technological Institute, Northwestern University, June 1968.

[10] Wilkinson, J. H., "The solution of ill-conditioned linear equations," Chapter 3 of Mathematical Methods for Digital Computers, vol. 2 (A. Ralston and H. S. Wilf, eds.), John Wiley & Sons, Inc., New York, 1967.

[11] Zlobec, S., and A. Ben-Israel, "On explicit solutions of interval linear programs," Systems Research Memo. No. 210, The Technological Institute, Northwestern University, July 1968.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Northwestern University	2a. REPORT SECURITY CLASSIFICATION Unclassified
	2b. GROUP Not applicable

3. REPORT TITLE

A DECOMPOSITION METHOD FOR INTERVAL LINEAR PROGRAMMING

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(Last name, first name, initial)*

Robers, Phillip D., and Ben-Israel, Adi

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1968	21	11

8a. CONTRACT OR GRANT NO. NSF Project GP 7550	9a. ORIGINATOR'S REPORT NUMBER(S) Systems Research Memorandum No. 216
b. PROJECT NO. Nonr-1228(10) Nooo14-67-A-0356-0006	
c. U. S. Army:	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d. Contract No. DA-31-124-ARO-D-322	

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Logistics and Mathematical Statistics Branch, Office of Naval Research

13. ABSTRACT

An interval linear program is

$$\text{(IP)} \quad \text{maximize } c^t x \text{, subject to } b^- \leqq Ax \leqq b^+$$

where the matrix A, vectors b^-, b^+, and c are given. If A has full row rank, the optimal solutions of (IP) can be written explicitly (A. Ben-Israel and A. Charnes: "An explicit solution of a special class of linear programming," Operations Research, forthcoming). This result is used in conjunction with the Dantzig-Wolfe decomposition principle to develop a finite iterative technique specially suited for solving the general (IP). Since any bounded linear program may be cast in form (IP) the technique may also be considered as an alternate method for linear programming.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
algorithm						
linear programming						
decomposition principle						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
____________________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
____________________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
____________________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1 JAN 64 1473 (BACK)

Unclassified
Security Classification